Subject: Partial Differential Equation
Subject Code: MMAT 203
M.Sc. Mathematics Second Semester

Classification of Partial Differential Equation

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Classification of Second Order Partial Differential Equation

The general form of a second order P.D.E is

$$\boldsymbol{A}\,(\boldsymbol{x},\boldsymbol{y})\,\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}^2}+\boldsymbol{B}\,(\boldsymbol{x},\boldsymbol{y})\,\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}\partial \boldsymbol{y}}+\boldsymbol{C}\,(\boldsymbol{x},\boldsymbol{y})\,\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{y}^2}+\boldsymbol{f}\,\left(\boldsymbol{x},\boldsymbol{y},\boldsymbol{u},\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{x}},\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{y}}\right)=\mathbf{0}\ \ldots\ldots\ldots..(\mathbf{1})$$

Where A, B, C are constants or continuous functions of x&y possessing

continuous partial derivatives and A is positive.

Equation (1) is said to be

- Elliptic , if $B^2 - 4AC < 0$
- Parabolic , if $B^2 - 4AC = 0$
- Hyperbolic , if $B^2 - 4AC > 0$

EXAMPLE -1

Classify the Partial Differential Equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

Solution:

Comparing the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with the general form of PDE

we get $A = 1 , B = 0 , C = 0$

$So\ ,\quad B^2 - 4AC = 0^2 - 4(1)(0) = 0$

Hence given P.D.E. is parabolic.

Solution of MCQ - 3

Classify the Partial Differential Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Solution:

Comparing the equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with the general form of PDE

we get $A = 1\,, B = 0\,, C = 1$

$So\,, \quad B^2 - 4AC = 0^2 - 4(1)(1) = -4 < 0$

Hence given P.D.E. is elliptic.

Solution of MCQ - 4

Classify the Partial Differential Equation $\frac{\partial^2 u}{\partial x^2}+3\frac{\partial^2 u}{\partial x \partial t}+\frac{\partial^2 u}{\partial t^2}=0$

Solution:

Comparing the equation $\frac{\partial^2 u}{\partial x^2}+3\frac{\partial^2 u}{\partial x \partial t}+\frac{\partial^2 u}{\partial t^2}=0$ with the general form of PDE

we get $A = 1$, $B = 3$, $C = 1$

So , $B^2 - 4AC = 3^2 - 4(1)(1) = 5 > 0$

Hence given P.D.E. is Hyperbolic.

EXAMPLE – 4 Classify the Partial Differential Equation:

$$ty_{tt} + 3y_{xt} + xy_{xx} + 17y_x + 2y_t = 0 \ldots\ldots\ldots..(1)$$

Solution :- $B^2 - 4AC = 9 - 4tx$, $A = x$, $B = 3$, $C = t$

Now Eq. ① is

Elliptic if $9 - 4tx < 0$

$\Rightarrow tx > \frac{9}{4}$

Parabolic if $9 - 4tx = 0$

$\Rightarrow tx = \frac{9}{4}$

Hyperbolic if $9 - 4tx > 0$

$\Rightarrow 9 > 4tx \Rightarrow tx < \frac{9}{4}$

Variable Separable Form

Example-1

Solve the Partial Differential Equation by using the method of Separation of Variables

$$\frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial t} - u = 0,\ u(x,0) = 6e^{-3x}$$

………………..(*)

Solution :- Assume the Trial Solution is

$$u = XT \longrightarrow (1)$$

where X is a function of x alone & T is a function of t only.

from (1), $\frac{\partial u}{\partial x} = X'T$ & $\frac{\partial u}{\partial t} = XT'$

Put these in (*)

$$X'T - 2XT' - XT = 0$$

$$\Rightarrow X'T = X(2T' + T)$$

Seperating the variables

$$\frac{X'}{X} = \frac{2T' + T}{T} = c \longrightarrow (2)$$

As X & T are independent variables, so (2) can be true only if each side is equal to some constant c,

$$\frac{X'}{X} = \frac{2T' + T}{T} = c \rightarrow (2)$$

As X & T are independent variables, so (2) can be true only if each side is equal to some constant c,

$$\frac{X'}{X} = c \rightarrow (3) \Rightarrow \log X = cx + \log c_1 \quad \text{(after integrating (3))}.$$

$$\Rightarrow X = c_1 e^{cx} \rightarrow (A)$$

& $$\frac{2T' + T}{T} = c \Rightarrow \frac{T'}{T} = \frac{1}{2}(c-1) \rightarrow (4)$$

$$\Rightarrow \log T = \frac{1}{2}(c-1)t + \log c_2 \quad \text{[after integrating (4)]}$$

$$\Rightarrow \log \frac{T}{c_2} = \frac{1}{2}(c-1)t \Rightarrow T = c_2 e^{\frac{1}{2}(c-1)t} \rightarrow (A)$$

from (1)

$$u = XT$$

Put (★★) & (★★★) in (1)

$$\therefore \quad u = (c_1 e^{cx})\left(c_2 e^{\frac{1}{2}(c-1)t}\right)$$

$$\Rightarrow \quad \boxed{u = c_1 c_2 e^{cx} e^{\frac{1}{2}(c-1)t} \longrightarrow \text{general Sol.}}$$

[Given] $\Rightarrow$ $\boxed{u(x,0) = 6e^{-3x}}$

Put this in general solution
$u(x,t) = c_1 c_2 e^{cx} e^{\frac{1}{2}(c-1)t}$

[from General Sol.] $\rightarrow$

$$u(x,0) = c_1 c_2 e^{cx} e^{0}$$

$$\Rightarrow \quad 6e^{-3x} = c_1 c_2 e^{cx}$$

$$\Rightarrow \quad \boxed{c_1 c_2 = 6} \quad \& \quad \boxed{c = -3}$$

{ as $e^{-3x} = e^{cx}$
$\Rightarrow c = -3$ }

So $\boxed{u = 6e^{-(3x+2t)}}$ [from general Sol.]

This is the Required Sol.

Assignment of Lecture-1

1. Classify the PDE's

(i) One Dimensional Heat Flow equation

$$a^2 u_{xx} = u_t$$

(ii) One Dimensional Wave equation

$$u_{tt} = a^2 u_{xx}$$

(iii) Two Dimensional Laplace Equation

$$u_{xx} + u_{yy} = 0$$

2. Use the Method of Separation of Variables to solve the equation

$$3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 \ , \qquad u(x,0) = 4e^{-x}$$

Answers:

1. (i) Parabolic

   (ii) Hyperbolic

   (iii) Elliptic

2. $u(x,y) = 4\, e^{\left(-x+\frac{3}{2}\, y\right)}$

Thank You